THE MAPLE PACKAGE FOR SL_2-INVARIANTS AND KERNEL OF WEITZENBÖCK DERIVATIONS.

LEONID BEDRATYUK

Abstract. We offer a Maple package `SL_2_Inv_Ker` for calculating of minimal generating sets for the algebras of joint invariants/semi-invariants of binary forms and for calculations of the kernels of Weitzenböck derivations.

1. Introduction

Let $V_d \cong \langle v_0, v_1, \ldots, v_d \rangle$ be $d+1$-dimensional SL_2-module of binary forms of degree d and let $V_{\boldsymbol{d}} = V_{d_1} \oplus V_{d_2} \oplus \cdots \oplus V_{d_n}$, $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$. Denote by $\mathcal{O}(V_{\boldsymbol{d}})^{SL_2}$ the algebra of polynomial SL_2-invariant functions on V. Denote by $\mathcal{O}(V_{\boldsymbol{d}})^{U_2}$ the algebra of polynomial U_2-invariant functions on V, where U_2 is the maximal unipotent subgroup of SL_2. We have the obvious inclusion $\mathcal{O}(V_{\boldsymbol{d}})^{SL_2} \subset \mathcal{O}(V_{\boldsymbol{d}})^{U_2}$. Moreover $\mathcal{O}(V_{\boldsymbol{d}})^{U_2} \cong \mathcal{O}(V_1 \oplus V_{\boldsymbol{d}})^{SL_2}$. The elements of the finitely generated algebras can be identified with the algebras of joint invariants and joint semi-invariants of the binary forms of degrees $\boldsymbol{d}$.

A linear derivation $\mathcal{D}$ of a polynomial algebra is called Weitzenböck derivation if its matrix (as linear map on the vector space generated by variables of the polynomial algebra) is nilpotent. Any Weitzenböck derivation $\mathcal{D}$ is completely determined by the Jordan normal form of its matrix. We will denote by $\mathcal{D}_{\boldsymbol{d}}$, $\boldsymbol{d} := (d_1, d_2, \ldots, d_n)$ the Weitzenbök derivation if its matrix consisting of n Jordan blocks of size d_1+1, d_2+1, $\ldots, d_n+1$, respectively.

As is well known, the kernel of the derivations $\mathcal{D}_{\boldsymbol{d}}$, $\boldsymbol{d} := (d_1, d_2, \ldots, d_n)$ is isomorphic to the algebra of joint semi-invariants (and to the algebra of covariants) for n binary forms of degrees $d_1, d_2, \ldots, d_n$. Thus, the calculations of a minimal generated sets of the algebras $\mathcal{O}(V_{\boldsymbol{d}})^{SL_2}$ or $\mathcal{O}(V_{\boldsymbol{d}})^{U_2}$ can be reduced to calculation of minimal generating set of the kernel of suitable Weitzenböck derivation. For instance, the algebra $\mathcal{O}(V_d)^{U_2}$ can be identified with $\ker \mathcal{D}$, where the derivation $\mathcal{D}$ is defined by $\mathcal{D}(v_i) = i v_{i-1}$, $i = 0, \ldots, d$.

2. Algorithm

The algebra $\mathcal{O}(V_{\boldsymbol{d}})^{U_2}$ is a finitely generated multigraded algebra under the multidegree-order:

$$\mathcal{O}(V_{\boldsymbol{d}})^{U_2} = (\mathcal{O}(V_{\boldsymbol{d}})^{U_2})_{\boldsymbol{m},0} + (\mathcal{O}(V_{\boldsymbol{d}})^{U_2})_{\boldsymbol{m},1} + \cdots + (\mathcal{O}(V_{\boldsymbol{d}})^{U_2})_{\boldsymbol{m},j} + \cdots ,$$

where each subspace $(\mathcal{O}(V_{\boldsymbol{d}})^{U_2})_{\boldsymbol{d},j}$ of joint semi-invariants of order j and multidegree $\boldsymbol{m} := (m_1, m_2, \ldots, m_n)$ is finite-dimensional. The formal power series

$$\mathcal{P}(\mathcal{O}(V_{\boldsymbol{d}})^{U_2}, z_1, z_2, \ldots, z_n, t) = \sum_{\boldsymbol{m},j=0}^{\infty} \dim((\mathcal{O}(V_{\boldsymbol{d}})^{U_2})_{\boldsymbol{m},j}) z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} t^j,$$

is called the multivariariate Poincaré series of the algebra of join semi-variants $\mathcal{C}_{\boldsymbol{d}}$. Note that each semi-invariant of order zero is an invariant.

Suppose that $\mathcal{P}(\mathcal{O}(V_{\boldsymbol{d}})^{SL_2}, t) = \dfrac{P(t)}{Q(t)}$ and $P(t)$, $Q(t)$ are coprime polynomials. Denote by $\beta_{\boldsymbol{d}}$ the degree of the denominator $Q(t)$. Many experimental data leads to the following conjecture:

Conjecture. A degree upper bound for irreducible invariants of the algebra $\mathcal{O}(V_{\boldsymbol{d}})^{SL_2}$ does not exceed $\beta_{\boldsymbol{d}}$.

In the first version of the package we used the following algorithm:

1. Calculate multivariate Poincaré series of the algebras $\mathcal{O}(V_{\boldsymbol{d}})^{SL_2}$ (or $\mathcal{O}(V_{\boldsymbol{d}})^{SL_2}$ or $\ker \mathcal{D}_{\boldsymbol{m}}$.)
2. For every term $z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} t^j$ of the Poincaré series calculate (by linear algebra method) a basis of the vector space of semi-invariants(or invariants or elements of kernel) and the multidegree $(m_1, m_2, \ldots, m_n)$.
3. Separate irreducible polynomials.
4. Stop calculation if $m_1 + m_2 + \ldots + m_n > 18$.

For the package procedures `Min_Gen_Set_Invariants_S` we use also the following simplified algorithm:

1. Calculate Poincaré series of the algebras $\mathcal{O}(V_{\boldsymbol{d}})^{SL_2}$.
2. For every term z^m of the Poincaré series calculate a basis of the vector space of invariants of the degree m.
3. Separate irreducible polynomials.
4. Stop calculation if $m > 18$.

The second algorithm works some fast for small values d_i and $n > 4$.

The package calculate the set of irreducible invariants up to degree $\min(18, \beta_{\boldsymbol{d}})$, but in all known computable cases this set coincides with a minimal generating set, see, for example, Brouwer's webpage http://www.win.tue.nl/∼aeb/math/invar/invarm.html

3. Installation.

The package can be downloaded from the web http://sites.google.com/site/bedratyuklp/.

(1) download the file `SL_2_Inv_Ker.mpl` and save it into your Maple directory.
(2) download the Xin's file (see link at the web page) `Ell2.mpl` and save it into your Maple directory.
(3) run Maple
(4) `> restart:  read "SL_2_Inv_Ker.mpl":read "Ell2.mpl":`
(5) If necessary use `> HelP();`

4. Package Procedures and Syntax

Procedure name: `Min_Gen_Set_Semi_Invariants`
Feature: Computes irreducible invariants of the algebra $\mathcal{O}(V_{\boldsymbol{d}})^{U_2}$ up to degree $\min(18, \beta_{\boldsymbol{d}})$.
Calling sequence: `Min_Gen_Set_Semi_Invariants`$([d_1, d_2, \ldots, d_n])$;
Parameters:

$[d_1, d_2, \ldots, d_n]$	-	a list of degrees of n binary forms.
n	-	an integer, $n \leq 11$.

Procedure name: `Min_Gen_Set_Invariants`
Feature: Computes irreducible invariants of the algebra $\mathcal{O}(V_{\boldsymbol{d}})^{SL_2}$ up to degree

$\min(18, \beta_{\boldsymbol{d}})$.
Calling sequence: `Min_Gen_Set_Invariants`$([d_1, d_2, \ldots, d_n])$;
Parameters:

$[d_1, d_2, \ldots, d_n]$	-	a list of degrees of n binary forms.
n	-	an integer, $n \leq 11$.

Procedure name: `Kernel_LLN_Der`
Feature: Computes irreducible elements of the kernel of Weitzenböck derivation $\mathcal{D}_{\boldsymbol{d}}$ up to degree $\min(18, \beta_{\boldsymbol{d}})$.
Calling sequence: `Kernel_LLN_Der`$([d_1, d_2, \ldots, d_n])$;
Parameters:

$[d_1, d_2, \ldots, d_n]$	-	a list of degrees of n binary forms.
n	-	an integer, $n \leq 11$.

Procedure name: `Min_Gen_Set_Invariants_S`
Feature: By using the second algorithm the procedure computes a set of irreducible invariants of the algebra $\mathcal{O}(V_{\boldsymbol{d}})^{SL_2}$ up to degree $\min(18, \beta_{\boldsymbol{d}})$.
Calling sequence: `Min_Gen_Set_Invariants_S`$([d_1, d_2, \ldots, d_n])$;
Parameters:

$[d_1, d_2, \ldots, d_n]$	-	a list of degrees of n binary forms.
n	-	an integer, $n \leq 11$.

5. Examples

5.1. **Compute $\mathcal{O}(V_4)^{SL_2}$.** Use the command

```
> dd:=[4]:INV:=Min_Gen_Set_Invariants(dd):
```

"calculating multivariate Poincare series...."
"done!, upper bound", 13
"———————————————— ——————"
"————————————degree———————————- –", 2
" irreducible invariant of multidegree ", [2], "found"
"————————————degree———————————- –", 3
" irreducible invariant of multidegree ", [3], "found"
"————————————degree———————————- –", 4
"————————————degree———————————- –", 5
"————————————degree———————————- –", 6
"————————————degree———————————- –", 7
"————————————degree———————————- –", 8
"————————————degree———————————- –", 9
"————————————degree———————————- –", 10
"————————————degree———————————- –", 11
"————————————degree———————————- –", 12
"————————————degree———————————- –", 13
"Total number of irreducible invariants in minimal generating set ", 2

To extract the two invariants from the set INV one should to use the commands

```
>INV[2];
```

$$\left\{6\,{x_2}^2+2\,x_0x_4-8\,x_1x_3\right\}$$

```
>INV[3];
```

$$\left\{-6\,{x_2}^3-6\,x_0{x_3}^2-6\,{x_1}^2x_4+6\,x_0x_2x_4+12\,x_1x_2x_3\right\}$$

5.2. **Compute $\mathcal{O}(V_3\oplus V_4)^{SL_2}$.** Use the command

```
> dd:=[3,4]:INV:=Min_Gen_Set_Invariants(dd):
```

"calculating multivariate Poincare series...."
"done!, upper bound", 13
"————————————————————————————————"
"——————————degree——————————", 2
" irreducible invariant of multidegree ", [0, 2], "found"
"——————————degree——————————", 3
" irreducible invariant of multidegree ", [0, 3], "found"
"——————————degree——————————", 4
" irreducible invariant of multidegree", [4, 0], "found"
"——————————degree——————————", 5
" irreducible invariant of multidegree ", [4, 1], "found"
" irreducible invariant of multidegree", [2, 3], "found"
"——————————degree——————————", 6
" irreducible invariant of multidegree", [4, 2], "found"
" irreducible invariant of multidegree", [4, 2], "found"
"——————————degree——————————", 7
" irreducible invariant of multidegree", [4, 3], "found"
" irreducible invariant of multidegree", [4, 3], "found"
" irreducible invariant of multidegree", [4, 3], "found"
"——————————degree——————————", 8
" irreducible invariant of multidegree ", [6, 2], "found"
" irreducible invariant of multidegree", [4, 4], "found"
"irreducible invariant of multidegree", [4, 4], "found"
"——————————degree——————————", 9
" irreducible invariant of multidegree", [6, 3], "found"
" irreducible invariant of multidegree", [6, 3], "found"
" irreducible invariant of multidegree", [6, 3], "found"
" irreducible invariant of multidegree", [4, 5], "found"
"——————————degree——————————", 10
" irreducible invariant of multidegree", [6, 4], "found"
" irreducible invariant of multidegree", [6, 4], "found"
"——————————degree——————————", 11
" irreducible invariant of multidegree", [6, 5], "found"
"——————————degree——————————", 12
"——————————degree——————————", 13
"Total number of invariants in a minimal generating set ", 20

To get the number of invariants of degree i use the command `nops(INV[i])`, for instance

```
> nops(INV[5]);
```

$$2$$

To extract and manipulate by these two invariants use the following commands

```
> Inv5_1:=INV[5][1]:Inv5_2:=INV[5][2];
```

$$Inv5\_2 := 6\,{x_1}^2{x_3}^2y_0 + 18\,{x_1}^2{x_2}^2y_2 - 12\,{x_1}^3x_3y_2 - 12\,{x_1}^3x_2y_3 + 6\,{x_0}^2{x_2}^2y_4 + 6\,{x_2}^4y_0 + 6\,{x_1}^4y_4 + 12\,{x_1}^2x_2x_3y_1 - 12\,x_0x_1{x_3}^2y_1 + 12\,x_0{x_2}^2x_3y_1 + 12\,x_0{x_1}^2x_3y_3 - 12\,x_0{x_1}^2x_2y_4 - 12\,{x_0}^2x_2x_3y_3 - 12\,x_1{x_2}^2x_3y_0 + 12\,x_0x_1{x_2}^2y_3 - 12\,x_0{x_2}^3y_2 + 6\,{x_0}^2{x_3}^2y_2 - 12\,x_1{x_2}^3y_1$$

5.3. **Compute $\mathcal{O}(V_1 \oplus V_1 \oplus V_2)^{U_2}$.** Use the command

```
> dd:=[1,1,2]:COV:=Min_Gen_Set_Semi_Invariants(dd):
```

"calculating multivariate Poincare series...."
"done!, upper bound", 13
"————————————————————————————"
"——————————degree——————————", 2
"irreducible semi-invariant of multidegree", [0, 0, 2], "and order ", 0, "found"
"irreducible semi-invariant of multidegree", [1, 1, 0], "and order ", 0, "found"
"irreducible semi-invariant of multidegree", [0, 1, 1], "and order ", 1, "found"
"irreducible semi-invariant of multidegree", [1, 0, 1], "and order ", 1, "found"
"——————————degree——————————", 3
"irreducible semi-invariant of multidegree", [2, 0, 1], "and order ", 0, "found"
"irreducible semi-invariant of multidegree", [1, 1, 1], "and order ", 0, "found"
"irreducible semi-invariant of multidegree", [0, 2, 1], "and order ", 0, "found"
"——————————degree——————————", 4
"——————————degree——————————", 5
"——————————degree——————————", 6
"——————————degree——————————", 7
"——————————degree——————————", 8
"——————————degree——————————", 9
"——————————degree——————————", 10
"——————————degree——————————", 11
"——————————degree——————————", 12
"——————————degree——————————", 13
"number of semi-invariant of minimal generating set ", 10

Below is this minimal generating set listed by degree

```
> COV[1];nops(%);
```

$$\{x_0, y_0, u_0\}$$
$$3$$

```
> COV[2];nops(%);
```

$$\{-x_0y_1 + x_1y_0, -y_1u_0 + y_0u_1, -x_1u_0 + x_0u_1, 2\,u_0u_2 - 2\,{u_1}^2\}$$
$$4$$

```
> COV[3];nops(%);
```

$$\{-2\,y_0y_1u_1+{y_1}^2u_0+{y_0}^2u_2,-x_1y_1u_0+x_1y_0u_1+x_0y_1u_1-x_0y_0u_2,\\ {x_1}^2u_0+{x_0}^2u_2-2\,x_0x_1u_1\}$$

3

5.4. **Compute the kernel of the Weitzenböck derivation $\mathcal{D}_{\boldsymbol{d}}$, $\boldsymbol{d}=(1,3)$.** Use the command

```
> dd:=[1,3]:Ker:=Kernel_LLN_Der(dd):
```

"calculating multivariate Poincare series...."
"done!, upper bound", 13
"——————————————————— —————————"
"———————————degree——————————-", 2
"irreducible element of multidegree", [1, 1], "and order ", 2, "found"
"irreducible element of multidegree", [0, 2], "and order ", 2, "found"
"———————————degree——————————-", 3
"irreducible element of multidegree", [2, 1], "and order ", 1, "found"
"irreducible element of multidegree", [0, 3], "and order ", 3, "found"
"irreducible element of multidegree", [1, 2], "and order ", 1, "found"
"———————————degree———————————", 4
"irreducible element of multidegree", [2, 2], "and order ", 0, "found"
"irreducible element of multidegree", [3, 1], "and order ", 0, "found"
"irreducible element of multidegree", [0, 4], "and order ", 0, "found"
"irreducible element of multidegree", [1, 3], "and order ", 2, "found"
"———————————degree——————————- –", 5
"irreducible element of multidegree", [2, 3], "and order ", 1, "found"
"———————————degree——————————- –", 6
"irreducible element of multidegree", [3, 3], "and order ", 0, "found"
"———————————degree——————————- –", 7
"———————————degree——————————- –", 8
"———————————degree——————————- –", 9
"———————————degree——————————- –", 10
"———————————degree——————————- –", 11
"———————————degree——————————- –", 12
"———————————degree——————————- –", 13
"number of semi-invariant in minimal generating set ", 13

```
> Ker[3];
```

$$\{-6\,{y_0}^2y_3+6\,y_0y_1y_2-2\,{y_1}^3,4\,x_1y_0y_2-6\,x_0y_0y_3+2\,x_0y_1y_2-2\,x_1{y_1}^2,-2\,{x_0}^2y_2-{x_1}^2y_0+2\,x_0x_1y_1\}$$

5.5. **Compute $\mathcal{O}(V_1\oplus V_1\oplus V_1\oplus V_2\oplus V_2)^{SL_2}$.** Use the command

```
> dd:=[1,1,1,2,2]:Inv:=Min_Gen_Set_Invariants_S(dd):
```

"calculating Poincare series...."
"done!, upper bound", 6
"——————————————————— —————————"

"——————————degree——————————- –", 2

6, "irreducible invariants found"

"——————————degree——————————- –", 3

12, "irreducible invariants found"

"——————————degree——————————- –", 4

" an irreducible invariant found"

" an irreducible invariant found"

" an irreducible invariant found"

" an irreducible invariant found"

" an irreducible invariant found"

" an irreducible invariant found"

"——————————degree——————————- –", 5

"——————————degree——————————- –", 6

"Total number of invariants", 24

```
> Ker[3];
```

$$\{3\,{y_0}^2v_2-6\,y_0y_1v_1+3\,{y_1}^2v_0, 3\,{y_0}^2w_2+3\,{y_1}^2w_0-6\,y_0y_1w_1, 6\,u_0u_1w_1-3\,{u_0}^2w_2-3\,{u_1}^2w_0,$$
$$-6\,y_0u_0v_2-6\,y_1u_1v_0+6\,y_1u_0v_1+6\,y_0u_1v_1, -6\,y_0u_0w_2-6\,y_1u_1w_0+6\,y_0u_1w_1+6\,y_1u_0w_1,$$
$$6\,u_0u_1v_1-3\,{u_1}^2v_0-3\,{u_0}^2v_2, 6\,u_1x_0v_1+6\,u_0x_1v_1-6\,u_0x_0v_2-6\,u_1x_1v_0,$$
$$6\,y_1x_0w_1-6\,y_1x_1w_0-6\,y_0x_0w_2+6\,y_0x_1w_1, -6\,u_1x_1w_0-6\,u_0x_0w_2+6\,u_0x_1w_1+6\,u_1x_0w_1,$$
$$6\,y_0x_1v_1-6\,y_1x_1v_0+6\,y_1x_0v_1-6\,y_0x_0v_2, -3\,{x_1}^2v_0+6\,x_0x_1v_1-3\,{x_0}^2v_2, -3\,{x_0}^2w_2+6\,x_0x_1w_1-3\,{x_1}^2w_0\}$$